Due Thurs

6.2 - Solutions About Ordinary Points

Consider the differential equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, which in standard form is $y'' + P(x)y' + Q(x)y = 0$.

Definition: A point $x = x_0$ is an **ordinary point** if P and Q are analytic at x_0 (meaning power series representations centered at $x = x_0$ exist). A point that is not ordinary is **singular**.

Theorem: If $x = x_0$ is an ordinary point of $a_2(x)y''+a_1(x)y'+a_0(x)y = 0$, then we can always find two linearly independent solutions of the form $y = \sum_{n=0}^{\infty} c_n(x-x_0)^n$.

for us, $x_0 = 0$

Example: Find two power series solutions of the given differential equation about the ordinary point $x = 0$.

$y'' + 2xy' + 2y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

combine

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} 2x\, n c_n x^{n-1} + \sum_{n=0}^{\infty} 2 c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k + \sum_{k=1}^{\infty} 2k c_k x^k + \sum_{k=0}^{\infty} 2 c_k x^k = 0$$

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots = a_0 + \sum_{n=1}^{\infty} a_n$$

$k=0 \quad k=0$

$$2c_2 + 2c_0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} + 2kc_k + 2c_k \right] x^k = 0$$

$$2c_2 + 2c_0 = 0 \Rightarrow c_2 = -c_0$$

$$(k+2)(k+1)c_{k+2} + (2k+2)c_k = 0$$

Solve for constant with highest subscript

$$\boxed{c_{k+2} = -\frac{2}{k+2} c_k, \quad k = 1, 2, 3}$$

Recurrence relation

$c_2 = -c_0$

$k=1 \quad c_3 = -\frac{2}{3} c_1$

$k=2 \quad c_4 = -\frac{1}{2} c_2 = \frac{1}{2} c_0$

$k=3 \quad c_5 = -\frac{2}{5} c_3 = \frac{4}{15} c_1$

$k=4 \quad c_6 = -\frac{1}{3} c_4 = \frac{1}{6} c_2 = -\frac{1}{6} c_0$

$k=5 \quad c_7 = -\frac{2}{7} c_5 = -\frac{8}{105} c_1$

Recall: $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

$$y = c_0 + c_1 x - c_0 x^2 - \frac{2}{3} c_1 x^3 + \frac{1}{2} c_0 x^4 + \frac{4}{15} c_1 x^5 + \cdots$$

General solution $y = c_0 \left(1 - x^2 + \frac{1}{2} x^4 + \cdots\right) + c_1 \left(x - \frac{2}{3} x^3 + \frac{4}{15} x^5 + \cdots\right)$

Form: $y = c_1 y_1 + c_2 y_2$

two power series solutions $y_1 = 1 - x^2 + \frac{1}{2} x^4 + \cdots, \quad y_2 = x - \frac{2}{3} x^3 + \frac{4}{15} x^5 + \cdots$

Example: Find two power series solutions of the given differential equation about the ordinary point $x = 0$.

$(x+2)y'' + xy' - y = 0$

$$(x+2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} x n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=2}^{\infty} 2n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n$$

$$-\sum_{n=0}^{\infty} c_n x^n = 0$$

- Make exponents match
- Make indices match

Exponents:

$$\sum_{k=1}^{\infty} (k+1)k\, c_{k+1} x^k + \sum_{k=0}^{\infty} 2(k+2)(k+1) c_{k+2} x^k$$

$$+ \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

Indices:

$$\underbrace{4c_2}_{k=0} - \underbrace{c_0}_{k=0} + \sum_{k=1}^{\infty}\left[k(k+1)c_{k+1} + 2(k+2)(k+1)c_{k+2} + (k-1)c_k\right]x^k = 0$$

recurrence relation

$$c_2 = \frac{1}{4}c_0, \quad \boxed{c_{k+2} = -\frac{k-1}{2(k+2)(k+1)}c_k - \frac{k}{2(k+2)}c_{k+1}, \quad k = 1,2,3,\ldots}$$

$k=1 \quad c_3 = -\frac{1}{6}c_2 = -\frac{1}{24}c_0$

$k=2 \quad c_4 = -\frac{1}{24}c_2 - \frac{1}{4}c_3$

$$= -\frac{1}{96}c_0 + \frac{1}{96}c_0 = 0$$

$$k=3 \quad c_5 = -\frac{1}{20}c_3 - \cancel{\frac{3}{10}c_4} = \frac{1}{480}c_0$$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$y = c_0 + c_1 x + \frac{1}{4}c_0 x^2 - \frac{1}{24}c_0 x^3 + \frac{1}{480}c_0 x^5 + \cdots$$

$$y = c_0\left(1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 + \cdots\right) + c_1 x$$

so
$$\boxed{\begin{aligned} y_1 &= 1 + \frac{1}{4}x^2 - \frac{1}{24}x^3 + \frac{1}{480}x^5 + \cdots \\ y_2 &= x \end{aligned}}$$

yes, I understand the process. 4/29 MH

Example: Find two power series solutions of the given differential equation about the ordinary point $x = 0$.

$y'' - (x+1)y' - y = 0$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

Exponents:
$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} (k+1)c_{k+1} x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

Indices:
$$\overset{k=0}{2c_2} - \overset{k=0}{c_1} - \overset{k=0}{c_0} + \sum_{k=1}^{\infty}\left[(k+2)\cancel{(k+1)}c_{k+2} - \cancel{(k+1)}c_{k+1} - \cancel{(k+1)}c_k\right]x^k = 0$$

$$c_2 = \frac{1}{2}c_1 + \frac{1}{2}c_0$$

$$\boxed{c_{k+2} = \frac{1}{k+2}c_{k+1} + \frac{1}{k+2}c_k, \quad k=1,2,3,\ldots}$$

Recurrence relation

$k=1 \quad c_3 = \frac{1}{3}c_2 + \frac{1}{3}c_1 = \frac{1}{6}c_1 + \frac{1}{6}c_0 + \frac{1}{3}c_1 = \frac{1}{2}c_1 + \frac{1}{6}c_0$

$k=2 \quad c_4 = \frac{1}{4}c_3 + \frac{1}{4}c_2$

$= \frac{1}{4}\left(\frac{1}{2}c_1 + \frac{1}{6}c_0\right) + \frac{1}{4}\left(\frac{1}{2}c_1 + \frac{1}{2}c_0\right)$

$= \frac{1}{4}c_1 + \frac{1}{6}c_0$

Start again

$$c_{k+2} = \frac{1}{k+2}c_{k+1} + \frac{1}{k+2}c_k, \quad k = 1, 2, 3, \ldots$$

Ignore c_1 (that is, let $c_1 = 0$)

$c_2 = \frac{1}{2}c_0$

$k=1 \quad c_3 = \frac{1}{3}c_2 + \cancel{\frac{1}{3}c_1}$

$= \frac{1}{6}c_0$

$k=2 \quad c_4 = \frac{1}{4}c_3 + \frac{1}{4}c_2$

$= \frac{1}{24}c_0 + \frac{1}{8}c_0 = \frac{1}{6}c_0$

Ignore c_0 (let $c_0 = 0$)

$c_2 = \frac{1}{2}c_1$

$c_3 = \frac{1}{3}c_2 + \frac{1}{3}c_1$

$= \frac{1}{6}c_1 + \frac{1}{3}c_1 = \frac{1}{2}c_1$

$c_4 = \frac{1}{4}c_3 + \frac{1}{4}c_2$

$= \frac{1}{8}c_1 + \frac{1}{8}c_1 = \frac{1}{4}c_1$

$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$

$y = c_0 + \frac{1}{2}c_0x^2 + \frac{1}{6}c_0x^3 + \cdots + c_1x + \frac{1}{2}c_1x^2 + \frac{1}{2}c_1x^3 + \cdots$

$y = c_0\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots\right) + c_1\left(x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots\right)$

$$\boxed{y_1 = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots, \quad y_2 = x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \cdots}$$

- give ≥3 terms per solution

Use an appropriate Maclaurin series to find two power series solutions of the given differential equation about the ordinary point $x = 0$.

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' + e^x y' - y = 0 + 0x + 0x^2 + 0x^3 + \cdots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \cdots$$

$$+c_1 + (c_1 + 2c_2)x + (\tfrac{1}{2}c_1 + 2c_2 + 3c_3)x^2 + (\tfrac{1}{6}c_1 + c_2 + 3c_3 + 4c_4)x^3 + \cdots$$

$$-c_0 - c_1 x - c_2 x^2 - c_3 x^3 - \cdots = 0$$

Const: $2c_2 + c_1 - c_0 = 0 \Rightarrow c_2 = -\tfrac{1}{2}c_1 + \tfrac{1}{2}c_0$

x: $2c_2 + 6c_3 = 0 \Rightarrow c_3 = -\tfrac{1}{3}c_2 = \tfrac{1}{6}c_1 - \tfrac{1}{6}c_0$

x^2: $\tfrac{1}{2}c_1 + c_2 + 3c_3 + 12c_4 = 0 \Rightarrow c_4 = -\tfrac{1}{24}c_1 - \tfrac{1}{12}c_2 - \tfrac{1}{4}c_3$

$$c_4 = -\tfrac{1}{24}c_1 + \tfrac{1}{24}c_1 - \tfrac{1}{24}c_0 - \tfrac{1}{24}c_1 + \tfrac{1}{24}c_0$$

$$c_4 = -\tfrac{1}{24}c_1$$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$y = c_0 + c_1 x + \left(-\tfrac{1}{2}c_1 + \tfrac{1}{2}c_0\right)x^2 + \left(\tfrac{1}{6}c_1 - \tfrac{1}{6}c_0\right)x^3 + \cdots$$

$$y = c_0\left(1 + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \cdots\right) + c_1\left(x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots\right)$$

$$\boxed{y_1 = 1 + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 + \cdots, \quad y_2 = x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots}$$